# A SIMPLE METHOD TO CORRECT THE REFLECTION ERROR OF ABSORBING BOUNDARY CONDITION IN THE FDTD ANALYSIS OF WAVEGUIDES

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**Abstract:** The recent progress in absorbing boundary conditions (ABCs), especially Berenger's perfectly matched layer (PML), promises to be very attractive for microwave circuit and packaging full–wave analysis using the FDTD method. However, in waveguides, multimodal and dispersive waves exist, which makes it difficult to minimize the error from absorbing boundary reflection. Several studies have shown that no single ABC, including the PML, is effective in absorbing energy having widely varying transverse distributions and group velocities. The possibility of excitation of evanescent modes by input spectral content below the cutoff frequency complicates the design of the ABC. In this paper, an efficient and simple method, the geometry rearrangement technique (GRT), is implemented to minimize the boundary truncation error in a waveguide. Numerical illustration of the propagation constant in a rectangular waveguide, terminated with Mur's first–order ABC, demonstrates the effectiveness of GRT in correcting the ABC–induced reflection error for waveguide problems.

## 1. Introduction

Shieded waveguide structures with slots, geometrical and/or material discontinuities have been of interest in integrated circuit and antenna applications over a long period. The FDTD method is considered to be appropriate for solving waveguide problems, especially if complex geometrical and material configurations are involved. In analyzing the EM wave propagation in waveguides, the FDTD technique approximates Maxwell's equations using finite differences over a computational domain truncated by ABCs. A major problem in such analysis is the accurate termination of the guided wave structure extending beyond the FDTD grid boundaries. The propagation in a waveguide can be multimodal and very dispersive, and the ABC used to terminate the waveguide must be able to absorb energy having widely varying transverse distributions and group velocities. The problem of evanescent mode excitation by a narrow pulse complicates the analysis. This makes it difficult to accurately implement the FDTD method in waveguide problems. Although recent advancements in ABC, such as Higdon's ABC [1] and the PML [2] significantly decrease the residual reflection in free space propagation, they are much less effective in waveguide problems. For example, a complicated hybrid formulation involving Higdon's ABC and several layers of PML has been shown to absorb energy only over select frequencies [3].

In this paper, we show that a very simple Mur's first-order ABC can be used effectively for waveguide analysis, because the boundary reflection error for propagating modes can be corrected accurately. We employ a superposition of two sub-problems, formulated by a geometrical rearrangement of the absorbing boundary, as described in [4] for microstrip transmission lines. Unlike the microstrip guided wave mode, the rectangular waveguide modes have varying transverse spatial distributions and smaller mode separation. Also, pulse distortion and dispersion are much more severe in waveguides than in microstrip problems. The correction procedure, termed as the geometry rearrangement technique (GRT), is very similar to how one cancels out reference plane calibration errors in waveguide material measurements by using two different sample lengths.

After introducing our excitation choice, we describe GRT and show how one can estimate the reflection from absorbing boundary. The next step is to use this reflection coefficient to correct the propagation constant in a rectangular waveguide. Although, for simplicity, we have chosen Mur's first-order ABC in the FDTD implementation, GRT can correct the error introduced by any ABC employed. The computed results for the waveguide indicate that the far-end longitudinal boundary can be located as close as 5 cells beyond the appropriate field sampling location, and yet, accurate results can be obtained when compared with the standard formula for the propagation constant. The

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conventional FDTD without GRT gives much less acurate solution.

## 2. Methodology

### 2.1. Excitation Choice

To implement the FDTD method in waveguide problems, we should choose an excitation which models the actual physical fields in spatial distribution and time dependence. Here, we employ the Hanning window function to alter the rising slope of the excitation from zero to steady state. This modified excitation has less of its spectrum located below the cutoff frequency and can arrive at steady state with minimal transient interference [5]. The monochromatic excitation for the dominant $TE_{10}$ mode in a rectangular waveguide, modified by the Hanning window, is given by

$$E_z = \begin{cases} \sin(\frac{\pi x}{W})\sin(2\pi f_m t)U(t), & T < t \\ \sin(\frac{\pi x}{W})\sin(2\pi f_m t)(\frac{1}{2} - \frac{1}{2}\cos(\frac{\pi t}{T}))U(t), & T > t \end{cases} \quad (1)$$

where $T$, the rise time, is chosen to be 10 cycles of the excitation frequency, $f_m$, $z$ and $x$ are transverse coordinates ($y$ is longitudinal) in the waveguide, $W$ is the larger dimension of the cross-section, and $U(t)$ is the Heaviside function. The transient waveform of this modified monochromatic excitation is shown in Fig. 1. It can be seen that the rise to steady state is gradual, which minimizes the spectrum located below the cutoff frequency. The pulse can be modified to any other mode by including the appropriate modal transverse distribution $f(x, z)$ instead of $\sin(\frac{\pi x}{W})$ in (1). If one does not know an approximate transverse field distribution, a 2D Laplace's equation can be solved for the transverse field subject to appropriate boundary conditions for the mode under consideration.

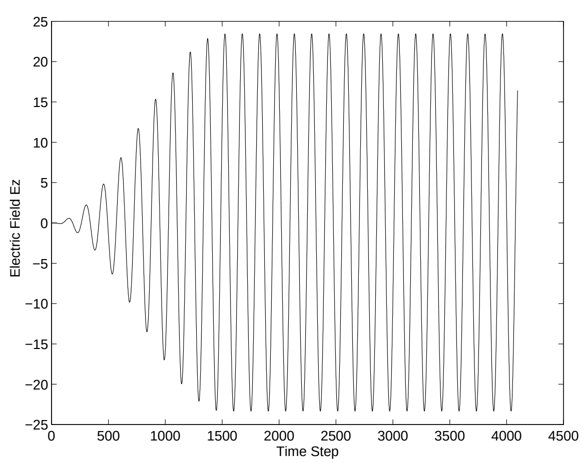


**Figure 1:** Transient waveform of a monochromatic excitation modified by the Hanning window [5].

### 2.2. Geometry Rearrangement Technique

The tangential field on the absorbing boundary, intended to simulate outgoing waves at the terminal planes of an FDTD grid, obeys the one-dimensional wave equation for propagation along the direction normal to the mesh wall. The wave will thus approach the end wall at normal incidence, with the dominant mode phase velocity, $v_p$. A commonly used first–order solution to the "one-way" wave equation is given by the Mur's ABC [6]:

$$E_z^n = E_{z-1}^{n-1} + \frac{v_p \delta t - \delta y}{v_p \delta t + \delta y}(E_z^{n-1} - E_{z-1}^n) \quad (2)$$

where $E_z$ represents the tangential electric field on the boundary, and $E_{z-1}$ represents the field a distance of one node inside the boundary. In waveguides, conventional FDTD implementation with the ABC in (2) introduces unacceptable error caused by reflection. Next, we show how GRT can correct such error.

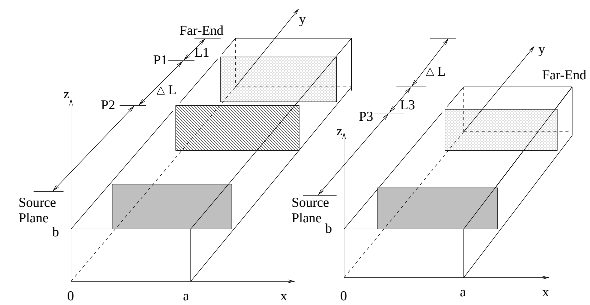


**Figure 2:** (a) Conventional field sampling, (b) far-end boundary moved closer by $\Delta L$.

In the conventional FDTD method, the frequency-dependent parameters such as the modal phase constant are calculated from the Fourier-transformed voltage (or electric field) at two different locations on a *single* waveguide. In order to reduce the influence of boundary reflection, this waveguide needs to be long enough such that, ideally, only forward traveling waves exist. While there is no unique definition for the voltage in single conductor waveguides, we compute voltage at the *i*-th port as the line integral over a cross-section (see Fig. 2):

$$V_i = \int_0^b E_z(\frac{a}{2}, y_i, z)dz. \quad (3)$$

With $V_1$ and $V_2$ denoting transforms of the FDTD-computed voltage at the points $P_1$ and $P_2$ (see Fig. 2), we have

$$e^{-\gamma(\omega)\Delta L} = \frac{V_1}{V_2} \quad (4)$$

where $\Delta L = L_2 - L_1$, $\gamma(\omega) = \alpha(\omega) + j\beta(\omega)$, $\beta(\omega) = \frac{1}{\Delta L}$ arctan $\angle\left[V_1/V_2\right]$, and $\omega$ is the angular frequency. *Note that $L_i$ is the distance of sampling point $P_i$, measured from the far–end boundary, and not the source plane.*

Eq. (4) neglects the reflection error induced by the imperfect ABC. We now examine how such error influences the computed propagation constant. We treat the far-end wall (Fig. 2) as a lumped load at the end of the waveguide, characterized by a frequency-dependent reflection coefficient $\Gamma_f$. Likewise, the reflection coefficient at the source-end boundary is $\Gamma_s$. The voltages $V_1$ and $V_2$ are then given by the superposition of longitudinally propagating incident wave and multiple reflections from source and far-end boundaries, and may be expressed as

$$V_1 = V_{1in} \frac{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_1}}{1 - \Gamma_f \Gamma_s e^{-2\tilde{\gamma}(\omega)L}} \tag{5}$$

$$V_2 = V_{2in} \frac{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_2}}{1 - \Gamma_f \Gamma_s e^{-2\tilde{\gamma}(\omega)L}} \tag{6}$$

where $V_{1in}$, $V_{2in}$ are incident voltages at $P_1$, $P_2$, respectively, $L$ is the length of the line between the boundaries (Fig. 2), and $\tilde{\gamma}(\omega)$ is the *true* propagation constant. From eqs. (4), (5) and (6), we obtain

$$e^{-\gamma(\omega)\Delta L} = \frac{V_{1in}}{V_{2in}} \frac{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_1}}{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_2}}. \tag{7}$$

The true propagation constant should be calculated from

$$e^{-\tilde{\gamma}(\omega)\Delta L} \equiv \frac{V_{1in}}{V_{2in}} \tag{8}$$

instead of (4) or (7), which are corrupted by boundary reflection. Eq. (8) follows from (7) if $L_1 = L_2$. How to realize this condition is the basis for GRT. We solve two identical problems with different boundary locations, as shown in Fig. 2, where the far–end boundary in the second problem (Fig. 2(b)) is brought closer to the source plane than in (Fig. 2(a)) by a distance $\Delta L$. This effectively replaces the sampling point $P_2$ with $P_3$ such that $V_{3in} = V_{2in}$, $L_3 = L_2 - \Delta L = L_1$. Then, using (8), we obtain

$$e^{-\tilde{\gamma}(\omega)\Delta L} = \frac{V_1}{V_3} \frac{1 - \Gamma_f \Gamma_s e^{-2\tilde{\gamma}(\omega)L}}{1 - \Gamma_f \Gamma_s e^{-2\tilde{\gamma}(\omega)(L-\Delta L)}} \frac{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_3}}{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_1}}. \tag{9}$$

Neglecting the composite reflection $\Gamma_f \Gamma_s$, which is small compared to unity (*e.g.*, see Fig. 3), and using $L_3 = L_1$, it follows that

$$e^{-\tilde{\gamma}(\omega)\Delta L} = \frac{V_1}{V_3}. \tag{10}$$

Thus, we have essentially sampled voltage $V_3$ instead of $V_2$ to obtain (8) from (7), and thereby correct the negative influence of dominant boundary reflection on calculation of the propagation constant. Accurate propagation constant can thus be obtained by keeping the boundary as close as 1 to 5 cell(s) from the far-end sampling location [4].

### 2.3. Calculating the Boundary Reflection

GRT can be used to estimate the boundary reflection caused by an imperfect ABC. With reference to Fig. 2, let $C = V_1/V_2$, and $G = V_1/V_3$, with $V_3$ calculated on the second line at $P_3$. Then, we obtain from (4), (5) and (6),

$$C = G \frac{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_1}}{1 + \Gamma_f e^{-2\tilde{\gamma}(\omega)L_2}} = G \frac{1 + \Gamma_f \, G^{\left(\frac{2L_1}{\Delta L}\right)}}{1 + \Gamma_f \, G^{\left(\frac{2L_2}{\Delta L}\right)}}. \tag{11}$$

Solving (11) for $\Gamma_f$, we obtain the reflection coefficient at the boundary

$$\Gamma_f = \frac{G - C}{G^{\left(\frac{2L_2}{\Delta L}\right)} C - G^{\left(\frac{2L_1}{\Delta L}+1\right)}}. \tag{12}$$

As an example, we consider a WR90 rectangular waveguide with $a = 22.86$ mm, $b = 10.16$ mm. Two waveguides are simulated, with one 120 cells long and the other 90 cells long. Both are terminated with Mur's first-order ABC on either end. The longer line is simulated to obtain $C$ and the shorter one, to calculate $G$. The cell dimensions are given by $\Delta x = 0.5715$ mm, $\Delta y = 0.3$ mm, and $\Delta z = 1.016$ mm (see Fig. 2 for definition of coordinates). The number of cells along $x$ and $z$ directions, respectively, is given by $N_x = 41$ and $N_z = 11$. The other dimensions are: $L_1 = L_3 = 10\Delta y$, $\Delta L = 30\,\Delta y$, and $L_2 = 40\,\Delta y$. The magnitude of the Mur's first-order boundary reflection coefficient, calculated from (12), is shown in Fig. 3.

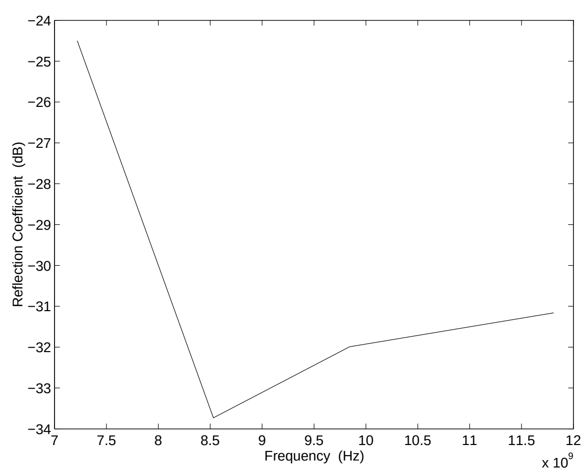


**Figure 3:** Reflection from absorbing boundary.

## 3. Simulation Results

To validate the afore-mentioned methodology, consider computation of the propagation constant in a uniform, rectangular waveguide assumed to be infinite along $y$ direction. The incident field is a modified monochromatic excitation of the form in (1) with $W = 22.86$ mm, and is applied 10 cells away from the $y = 0$ plane. The computed phase constant as a function of frequency is plotted in Fig. 4 against a reference solution calculated from the well-known analytical formula. The conventional FDTD implementation with imperfect ABC causes boundary reflection, which translates to significant numerical error (over 6%). The GRT result is quite smooth and agrees very well (within 1%) with the formula. The GRT/FDTD implementation employs $\Delta L = 30$ cells and first-order Mur's ABC on the two longitudinal boundaries.

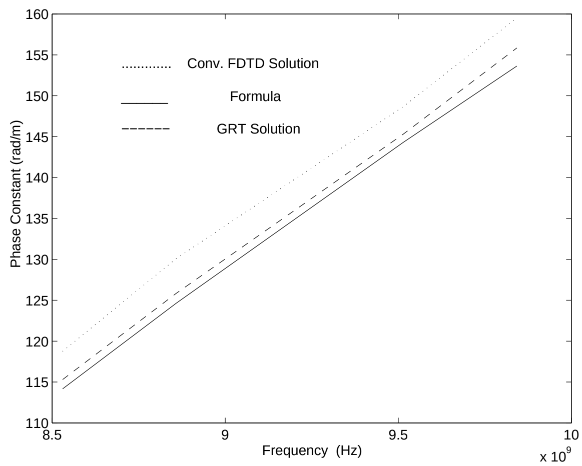


**Figure 4:** Phase constant of the dominant mode.

## 4. Summary

In this report, we have applied GRT to reduce the influence of boundary reflection in the analysis of wave propagation in a rectangular waveguide. It has been shown that the accuracy of numerical calculation of the propagation constant can be improved by correcting for the reflection error ensuing from an imperfect boundary condition. GRT involves the solution of two sub-problems differing in geometry only in the position of the far-end longitudinal boundary. The computational requirements of GRT are less than those of the conventional FDTD implementation, in which the absorbing boundary needs to be placed far away from the scattering element to reduce the boundary-scatterer interaction. The real utility of GRT is in the analysis of waveguide discontinuities and slot radiators. In this case, by solving two problems differing only in the position of the longitudinal boundary, it is possible to correct the slot admittance, or the S-parameters of the discontinuity, for the ABC-induced reflection error. The theory for effecting such correction in a discontinuity problem is described in [7], [8].

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